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ON THE MULTIPLE SCATTERING OF WAVES FROM OBSTACLES WITH SOLID-F--ETC(U)

JUL 79 B PETERSON, V V VARADAN, V K VARADAN

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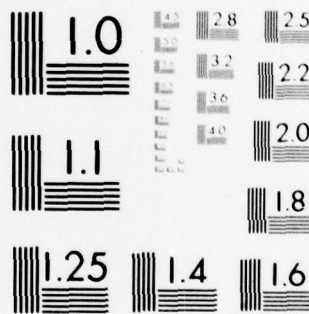
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ON THE MULTIPLE SCATTERING OF WAVES FROM OBSTACLES WITH  
SOLID-FLUID INTERFACES,

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Bo Peterson, V.V. Varadan and V.K. Varadan  
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Rept. for Oct 78-31 Jul 79

For the Period  
October 1, 1978 - July 31, 1979

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DEPARTMENT OF THE NAVY  
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Contract No. N00014-78-C-0559

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ON THE MULTIPLE SCATTERING OF WAVES FROM OBSTACLES WITH  
SOLID-FLUID INTERFACES

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I. INTRODUCTION

The T-matrix or null field method has proved to be an efficient computational scheme for analyzing the scattering of waves for several difficult geometries. Its broad applicability to many fields of engineering and science in handling acoustic, quantum mechanical, electro-magnetic or -static, and elastic problems has been realized only recently, see (1-16). The T-matrix approach has also been extended to treat increasingly complicated situations such as multilayered scatterers (17-25), multiple scattering by a finite number of scatterers (19-25), multiple scattering in lattices (26), and multiple scattering by statistical distribution of scatterers (27). However, in the theory of elasticity the analysis of structures involving several layers with solid-fluid interfaces is still lacking. The purpose of this article is to present a T-matrix formalism for elastic wave scattering by multilayered regions with solid-fluid interfaces. Here the layers are not necessarily consecutively enclosing and thus in some regions there may be several different scatterers. The basic methods for this extension are given in (21-25). We are using the Green's dyadic in an integral representation although the T-matrix can be obtained without using any Green's dyadic. (8).

This article consists of 9 parts. Here, in part I, we introduce the basic problems to be treated. In part II we obtain the integral representations for a displacement field in the case of layered regions. Then, we introduce the spherical wave functions and use them to expand the Green's dyadic. In part III we expand the various fields in spherical wave functions. Using the integral representations derived in part II we obtain the Q-matrix for the outermost surface of a body separating two elastic materials. This matrix and the T-matrix associated with the body inside the next outermost surface form the total T-matrix for the body. The procedure in part IV is similar to part III; however, we must now deal with a fluid outside and an elastic material inside the outermost surface. This new situation requires the defining of three matrices (the P, Q, and R matrices) for the outermost surface, in contrast to the simpler, "single surface matrix" case of part III. These three matrices and the T-matrix associated with the body inside the next outermost surface form the total T-matrix for the body. In part V we have an elastic material outside and a fluid inside the outermost surface of the body. Here we have to introduce the U, V, W and Y matrices associated with the outermost surface. These four matrices, together with the T-matrix associated with the next outermost surface of the two layered body, form the total T-matrix for the body.

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In part VI we study scattering from several nonoverlapping bodies. In part VII we use the results from the parts II-VI to get a general procedure for obtaining the T-matrix associated with a multilayered, finite body. These layers may be either elastic materials or fluids, and need not be successively enclosing. In part VIII we review the terminology for scattering data. In part IX we give numerical results for rotationally symmetric bodies in a fluid. The wavelengths under consideration are in the longwave- and resonance-regions. We also consider complex wave numbers, the moduli of which are less than the inverse power of some characteristic body dimension. The results are presented in the form of frequency dependent plots for the bistatic differential scattering cross section, the total scattering cross section, and the absorption cross section. We also present some polar plots of the differential scattering cross section.

## II. INTEGRAL REPRESENTATIONS AND SPHERICAL WAVE FUNCTIONS

Consider an infinite, homogeneous elastic material with density  $\rho_0$  and Lamé parameters  $\lambda_0$  and  $\mu_0$ . A finite layered body is introduced in this material. The body is such that between the outer bounding surface  $S_1$  and an inner surface  $S_2$  the density is  $\rho_1$  and the Lamé parameters  $\lambda_1$  and  $\mu_1$ . The material parameters inside the surface  $S_2$  can be varying with the space coordinates in an arbitrary way. The region outside the surface  $S_1$  and the region between the surfaces  $S_1$  and  $S_2$  are called region 0 and region 1, respectively. We assume that an origin can be chosen inside the two surfaces in such a way that, for each surface, the radius (from the origin) to that surface is a continuous single-valued function of the spherical angles. Furthermore, we assume that both surfaces satisfy the requirements of the divergence theorem. Let a monochromatic displacement field  $\vec{u}_0^i$ , with time dependence  $\exp(-i\omega t)$ , impinge on the body. The scattered field  $\vec{u}_0^s$  and the incoming field  $\vec{u}_0^i$  give the total field in region 0 as:

$$\vec{u}_0 = \vec{u}_0^i + \vec{u}_0^s \quad (2.1)$$

The total field between the surface  $S_1$  and  $S_2$  (region 1) is called  $\vec{u}_1$ . Then with the exception of source points we have the following equations of motion for  $\vec{u}_1(\vec{r})$ :

$$\frac{\kappa_1^3}{k_1^2} \nabla \nabla \cdot \vec{u}_1 - \kappa_1 \nabla \times (\nabla \times \vec{u}_1) + \kappa_1^3 \vec{u}_1 = 0 \quad (2.2)$$

$$\text{where } i = 0 \text{ or } 1 \text{ and } k_1^2 = \frac{\rho_1 \omega^2}{\lambda_1 + 2\mu_1} \text{ and } \kappa_1^2 = \frac{\rho_1 \omega^2}{\mu_1}.$$

For the same regions we define the Green's dyadic  $\vec{G}_1(k_1, \kappa_1, |\vec{r} - \vec{r}'|)$  as the outgoing solution of

$$\frac{\kappa_1^3}{k_1^2} \nabla \nabla \cdot \vec{G}_1 - \kappa_1 \nabla \times (\nabla \times \vec{G}_1) + \kappa_1^3 \vec{G}_1 = -\vec{I} \delta(\vec{r} - \vec{r}'), \quad (2.3)$$

where  $i = 0$  or  $1$  and  $\vec{I}$  is the unit dyadic and  $\delta$  is the Dirac distribution. The stress-dyadics and triadics for the same regions are given by (2.4) and (2.5), respectively.

$$\vec{\sigma}_1(\vec{u}) = \lambda_1 \vec{I} \nabla \cdot \vec{u} + \mu_1 (\nabla \vec{u} + \vec{u} \nabla) \quad (2.4)$$

$$\vec{\xi}_1(\vec{G}) = \lambda_1 \vec{I} \nabla \cdot \vec{G} + \mu_1 (\nabla \vec{G} + \vec{G} \nabla) \quad (2.5)$$

where  $i = 0$  or  $1$ , and  $\vec{u}$  and  $\vec{G}$  are the arbitrary displacement field and Green's dyadic, respectively. The surface tension  $t_1$  is related to the stress dyadic by the surface normal  $\hat{n}$ .

$$\vec{t}_1 = \hat{n} \cdot \vec{\sigma}_1 \quad (2.6)$$

The equations (2.2) and (2.3) can be rewritten in terms of the stress-dyadic and triadic, respectively.

$$\nabla \cdot \vec{\sigma}_1(\vec{u}_1) + \rho_1 \omega^2 \vec{u}_1 = 0 \quad (2.2')$$

$$\nabla \cdot \vec{\xi}_1(\vec{G}_1) + \rho_1 \omega^2 \vec{G}_1 = - \vec{I} \frac{\rho_1 \omega^2}{\kappa_1} \delta(\vec{r} - \vec{r}') \quad (2.3')$$

These latter forms may be better suited for the derivation of the integral representation. Here equation (2.2') is expressed in the same terms, which appear in the boundary conditions of the integral representation. However, the earlier forms ((2.2) and (2.3)) may be better suited for obtaining the basis functions or Green's dyadic. Here we can clearly see how to separate the curl free and divergence free parts. By using equations (2.2), (2.3), and the divergence theorem we can obtain the integral representations for the various regions (9).

$$\vec{u}_0^1(\vec{r}) + \frac{\kappa_0^3}{\rho_0 \omega^2} \int_{S_1} [\vec{u}_{0+} \cdot (\hat{n}_1 \cdot \vec{\xi}_0) - \hat{n}_1 \cdot \vec{\sigma}_{0+} \cdot \vec{G}_0] ds' = \begin{cases} \vec{u}_0(\vec{r}), & \vec{r} \text{ outside } S_1. \\ 0, & \vec{r} \text{ inside } S_1. \end{cases} \quad (2.7)$$

$$\begin{aligned} & - \frac{\kappa_1^3}{\rho_1 \omega^2} \int_{S_1} [\vec{u}_{1-} \cdot (\hat{n}_1 \cdot \vec{\xi}_1) - \hat{n}_1 \cdot \vec{\sigma}_{1+} \cdot \vec{G}_1] ds' + \\ & + \frac{\kappa_1^3}{\rho_1 \omega^2} \int_{S_2} [\vec{u}_{1+} \cdot (\hat{n}_2 \cdot \vec{\xi}_1) - \hat{n}_2 \cdot \vec{\sigma}_{1+} \cdot \vec{G}_1] ds' = \begin{cases} \vec{u}_1(\vec{r}), & \vec{r} \text{ between } S_1 \text{ and } S_2. \\ 0, & \vec{r} \text{ outside } S_1 \text{ or inside } S_2. \end{cases} \end{aligned} \quad (2.8)$$

Here, the + and - subscripts stand for the limit values calculated from the outside and the inside of the surface, respectively. If the material in one of the regions was a fluid ( $\mu_1 = 0$  for some  $i$ ), we would get (2.9) and (2.10) instead of (2.2) and (2.3).

$$\nabla \nabla \cdot \vec{u}_1 + \kappa_1^2 \vec{u}_1 = 0 \quad (2.9)$$

$$\nabla \nabla \cdot \vec{C}_1 + k_1^2 \vec{C}_1 = -\frac{1}{k_1} \vec{I} \delta(\vec{r} - \vec{r}') \quad (2.10)$$

The only change in the integral representations (2.7) and (2.8) is that we have to change the  $\kappa_1$  to a  $k_1$ .

Alternatively, when dealing with a fluid we can work with a scalar potential  $\phi_1$ , which is related to the displacement  $\vec{u}_1$  by:

$$\vec{u}_1 = \nabla \phi_1. \quad (2.11)$$

The potential  $\phi_1(\vec{r})$  and the corresponding Green's function  $g_1(k_1, |\vec{r} - \vec{r}'|)$  have the following equations of motion:

$$(\nabla^2 + k_1^2) \phi_1 = 0 \quad (2.12)$$

$$(\nabla^2 + k_1^2) g_1 = -\delta(\vec{r} - \vec{r}') \quad (2.13)$$

From the equations (2.12) and (2.13) we can derive the following integral representations valid only for fluids.

$$\phi_0^1(\vec{r}) + \int_{S_1} [\phi_{0+} \nabla' g_0 - (\nabla' \phi_{0+}) g_0] \cdot d\vec{s}' = \begin{cases} \phi_0(\vec{r}), & \vec{r} \text{ outside } S_1. \\ 0, & \vec{r} \text{ inside } S_1. \end{cases} \quad (2.14)$$

$$- \int_{S_1} [\phi_{1-} \nabla' g_1 - (\nabla' \phi_{1-}) g_1] \cdot d\vec{s}' + \int_{S_2} [\phi_{1+} \nabla' g_1 - (\nabla' \phi_{1+}) g_1] \cdot d\vec{s}' = \begin{cases} \phi_1(\vec{r}), & \left\{ \begin{array}{l} \vec{r} \text{ between} \\ S_1 \text{ and } S_2. \end{array} \right. \\ 0, & \left\{ \begin{array}{l} \vec{r} \text{ outside } S_1 \\ \text{or} \\ \text{inside } S_2. \end{array} \right. \end{cases} \quad (2.15)$$

Here  $\phi_0$ , the total field in region 0, is given in terms of the incoming field  $\phi_0^1$  and scattered field  $\phi_0^s$  by:

$$\phi_0 = \phi_0^1 + \phi_0^s. \quad (2.16)$$

To use the above integral representations (2.7), (2.8), (2.14) or (2.15) one must introduce boundary conditions. These may be expressed in terms of physical observables such as displacement and tension at the surface  $S_1$ . The surface tension  $\vec{t}_1$  is expressible in a general material as a function of the displacement  $\vec{u}$  as seen from equations (2.4) and (2.6). In the particular case of a fluid the surface tension  $\vec{t}_{0+}$  can be expressed as a function of the potential  $\phi_0$  as:

$$\vec{t}_{0+} = -k_0^2 \lambda_0 \phi_{0+} \hat{n}_1 = -\rho_0 \omega^2 \phi_{0+} \hat{n}_1. \quad (2.17)$$



In the case of a fluid, either of the equations (2.7) or (2.14) as well as either of the equations (2.8) or (2.15) can be derived from the other.

It is our aim to develop a matrix representation of the fields. The spherical wave solutions (2.18)-(2.20) to eq. (2.2) are well suited as a basis in such a representation.

$$\psi_{1\sigma mn}^1(\vec{r}) = \frac{1}{\sqrt{n(n+1)}} \nabla_x [\vec{r} Y_{\sigma mn}(\theta, \phi) h_n(\kappa_1 r)] \quad (2.18)$$

$$\psi_{2\sigma mn}^1(\vec{r}) = \frac{1}{\kappa_1} \nabla_x \psi_{1\sigma mn}^1(\vec{r}) \quad (2.19)$$

$$\psi_{3\sigma mn}^1(\vec{r}) = \left(\frac{\kappa_1}{\kappa_1}\right)^{3/2} \frac{1}{\kappa_1} \nabla [Y_{\sigma mn}(\theta, \phi) h_n(\kappa_1 r)] \quad (2.20)$$

$$Y_{\sigma mn}(\theta, \phi) = \left[ \epsilon_m \frac{(2n+1)(n-m)!}{4\pi(n+m)!} \right]^{1/2} P_n^m(\cos\theta) \begin{cases} \cos m\phi, & \sigma = e \\ \sin m\phi, & \sigma = o \end{cases} \quad (2.21)$$

$$= Y_{mn}^m P_n^m(\cos\theta) \begin{cases} \cos m\phi \\ \sin m\phi \end{cases},$$

where  $\epsilon_0 = 1$  and  $\epsilon_m = 2$  for  $m > 0$ ,  $e$  and  $o$  stands for even resp. odd,  $h_n$  is the (outgoing) spherical Hankel function,  $P_n^m$  is the associated Legendre polynomial. We are going to use various abbreviations  $\psi_{1\sigma mn}^1 \equiv \psi_{1p}^1 \equiv \psi_q^1$ . Further, the symbols  $Ou$  and  $Re$  will represent the outgoing and regular functions, respectively. For example,  $Ou\psi_q^1 = \psi_q^1$ , but, in contrast,  $Re\psi_q^1$  means that, instead of using  $h_n$ , we use the regular function  $j_n$  (at the origin). This is not necessarily the same as taking the real part of  $\psi_q^1$ . The spherical wave solution to eq. (2.12) is:

$$\phi_{\sigma mn}^1(\vec{r}) = Y_{\sigma mn}(\theta, \phi) h_n(\kappa_1 r). \quad (2.22)$$

When dealing with a fluid we let  $\psi_{1p}^1 = \psi_{2p}^1 = 0$  and remove the factor  $(\kappa_1/\kappa_1)^{3/2}$  in  $\psi_{3p}^1$ . Given the above special conditions we can interpret the expansion of the Green's dyadic  $\vec{G}_1$  in spherical wave functions for all cases as:

$$\vec{G}_1(k_1, \kappa_1, |\vec{r} - \vec{r}'|) = i \vec{r}_> \psi_q^1(\vec{r}_>) Re\psi_q^1(\vec{r}_<) , \quad (2.23)$$

where  $\vec{r}_>$  and  $\vec{r}_<$  stands for the greater resp. smaller of  $\vec{r}$  and  $\vec{r}'$ . The Green's function  $g_1$  can be expanded as:

$$g_1(k_1, |\vec{r} - \vec{r}'|) = i k_1 \vec{r}_p \phi_p^1(\vec{r}_>) Re\phi_p^1(\vec{r}_<) . \quad (2.24)$$

### III. LAYERED BODIES WITH ELASTIC MATERIAL ON BOTH SIDES OF THE BOUNDING SURFACE

Consider an elastic body, with an inclusion, situated in an infinite elastic material. Outside the bounding surface  $S_1$  as well as between  $S_1$  and an inner surface  $S_2$ , the material parameters are constant and given in the notations from section II. In section II we introduced the spherical vector wave functions as a basis and expanded the Green's dyadic in this basis. Here we expand the



displacement fields in the same basis. The expansion of the incoming and scattered field (in the infinite material) is given by (3.1) and (3.2), respectively.

$$\vec{u}_0^i(\vec{r}) = \sum_q a_q \text{Re} \vec{\psi}_q^0(\vec{r}) \text{ for } r < r^{so} \quad (3.1)$$

$$\vec{u}_0^s(\vec{r}) = \sum_q f_q \vec{\psi}_q^0(\vec{r}) \text{ for } r > r_{\max}^1 \quad (3.2)$$

Here,  $r^{so}$  is the length of the radius vector from origin to the closest singularity of the sources of the incoming field. The symbol  $r_{\max}^1$  is the radius of the sphere  $X_1$ , with center at the origin, circumscribing  $S_1$  (see Fig. 1). By using the integral representation (2.7) and considering  $\vec{r}$  outside the sphere  $X_1$ , applying the expansions (3.2) and (2.23), we get (3.3), after identifying coefficients.

$$f_q = i \frac{\kappa_0^3}{\rho_0 \omega^2} \int_{S_1} [\vec{t}_0(\text{Re} \vec{\psi}_q^0) \cdot \vec{u}_{0+} - \text{Re} \vec{\psi}_q^0 \cdot \vec{t}_{0+}(\vec{u}_0)] ds \quad (3.3)$$

Let  $Z_1$  be the sphere, with center at the origin, inscribed in  $S_1$  (see Fig. 1). Considering  $\vec{r}$  inside  $Z_1$  we get (3.4), in a similar way as above, from (2.7).

$$a_q = -i \frac{\kappa_0^3}{\rho_0 \omega^2} \int_{S_1} [\vec{t}_0(\vec{\psi}_q^0) \cdot \vec{u}_{0+} - \vec{\psi}_q^0 \cdot \vec{t}_{0+}(\vec{u}_0)] ds \quad (3.4)$$

The total T-matrix (32-34) for the layered body, defined by (3.5), can not yet be obtained because we have not yet specified the properties of the body.

$$f_q = \sum_q T_{qq} a_q \quad (3.5)$$

The relations (3.3) and (3.4) together constitute a formal definition of the T-matrix through the two mappings from the surface field to the scattered field and the incoming field, respectively. To realize these mappings we introduce the boundary conditions at  $S_1$  and expand the field on the inside of  $S_1$ . The welded boundary conditions (3.6) and (3.7) relate the limit values of the field quantities from each side of the surface  $S_1$ .

$$\vec{u}_{0+} = \vec{u}_{1-} \text{ (for } \vec{r} \in S_1 \text{)} \quad (3.6)$$

$$\vec{t}_{0+} = \vec{t}_{1-} \text{ (for } \vec{r} \in S_1 \text{)} \quad (3.7)$$

The expansion of the field on the inside of  $S_1$  is given by:

$$\vec{u}_{1-} = \sum_q (\alpha_q^1 \text{Re} \vec{\psi}_q^1 + \beta_q^1 \vec{\psi}_q^1) \quad (3.8)$$

The completeness of expansions of this kind has been treated in (28-31). Using (3.6), (3.7), and (3.8) in (3.3) and (3.4), we get:

$$f_q = i \sum_q (\text{Re} \text{Re} Q_{qq}^1 \alpha_q^1 + \text{Re} O u Q_{qq}^1 \beta_q^1) \quad (3.9)$$

and

$$a_q = -i \Sigma_q (OuReQ_{qq}^1, \alpha_q^1 + OuOuQ_{qq}^1, \beta_q^1), \quad (3.10)$$

where the matrix  $Q^1$  is given by

$$\begin{pmatrix} Ou \\ Re \end{pmatrix} \begin{pmatrix} Ou \\ Re \end{pmatrix} Q_{qq'}^{i+1} = \frac{\kappa_1^3}{\rho_1 \omega^2} \int_{S_{i+1}} \left[ \vec{t}_1 \left( \begin{pmatrix} Ou \\ Re \end{pmatrix} \vec{\psi}_q^i \right) \cdot \begin{pmatrix} Ou \\ Re \end{pmatrix} \vec{\psi}_{q'}^{i+1} - \right. \\ \left. \begin{pmatrix} Ou \\ Re \end{pmatrix} \vec{\psi}_q^{i+1} \cdot \vec{t}_{i+1} \left( \begin{pmatrix} Ou \\ Re \end{pmatrix} \vec{\psi}_{q'}^{i+1} \right) \right] ds \quad \text{for } i = 0. \quad (3.11)$$

Here, the first  $\begin{pmatrix} Ou \\ Re \end{pmatrix}$  in the left member of equation (3.11) corresponds to the selection of either the regular or outgoing functions associated with the first index in  $Q^1$ . Similarly, the second  $\begin{pmatrix} Ou \\ Re \end{pmatrix}$  is related to the second index. In vector matrix notation we rewrite (3.9) and (3.10) as:

$$\vec{f} = iReReQ_{\alpha}^{1+1} + iReOuQ_{\beta}^{1+1} \quad (3.12)$$

$$\vec{a} = -iOuReQ_{\alpha}^{1+1} - iOuOuQ_{\beta}^{1+1} \quad (3.13)$$

If the body under consideration is homogeneous (no inclusion) which, in turn, implies that  $\vec{\beta}^1 \equiv 0$  we can solve the equations (3.12) and (3.13) by elimination of  $\alpha^1$ . In this special case we get the T-matrix for a homogeneous elastic body as:

$$T = -ReReQ^1 (OuReQ^1)^{-1}. \quad (3.14)$$

However, if the body has an inclusion, with bounding surface  $S_2$ , we have to introduce the integral representation (2.8). By using the integral representation (2.8) and considering  $\vec{r}$  outside the sphere  $X_1$  we get (3.15). Further, by using (2.8) and considering  $\vec{r}$  inside the sphere  $Z_2$  we get (3.16). The sphere  $Z_2$  is defined for the surface  $S_2$  in a way similar to the defining of the sphere  $Z_1$  with respect to the surface  $S_1$  (see Fig. 1).

$$\Sigma_q (ReReX_{qq}^1, \alpha_q^1 + ReOuX_{qq}^1, \beta_q^1) - \frac{\kappa_1^3}{\rho_1 \omega^2} \int_{S_2} [\vec{t}_1 (Re\vec{\psi}_q^1) \cdot \vec{u}_{1+} - \\ - Re\vec{\psi}_q^1 \cdot \vec{t}_{1+}(\vec{u}_1)] ds = 0, \quad (3.15)$$

$$\Sigma_q (OuReX_{qq}^1, \alpha_q^1 + OuOuX_{qq}^1, \beta_q^1) - \frac{\kappa_1^3}{\rho_1 \omega^2} \int_{S_2} [\vec{t}_1 (\vec{\psi}_q^1) \cdot \vec{u}_{1+} - \\ - \vec{\psi}_q^1 \cdot \vec{t}_{1+}(\vec{u}_1)] ds = 0, \quad (3.16)$$

where

$$\begin{aligned} \left\{ \begin{matrix} \text{Ou} \\ \text{Re} \end{matrix} \right\} \left\{ \begin{matrix} \text{Ou} \\ \text{Re} \end{matrix} \right\} X_{qq'}^1 &= \frac{\kappa_1^3}{\rho_1 \omega^2} \int_{S_1} \left[ \vec{t}_1 \left( \left\{ \begin{matrix} \text{Ou} \\ \text{Re} \end{matrix} \right\} \vec{\psi}_q^{+1} \right) \cdot \left\{ \begin{matrix} \text{Ou} \\ \text{Re} \end{matrix} \right\} \vec{\psi}_q^{+1} \right. \\ &\quad \left. - \left\{ \begin{matrix} \text{Ou} \\ \text{Re} \end{matrix} \right\} \vec{\psi}_q^{+1} \cdot \vec{t}_1 \left( \left\{ \begin{matrix} \text{Ou} \\ \text{Re} \end{matrix} \right\} \vec{\psi}_q^{+1} \right) \right] ds \quad \text{for } i = 1. \end{aligned} \quad (3.17)$$

It is a straight forward matter to show (3.18).

$$\left\{ \begin{matrix} \text{Ou} \\ \text{Re} \end{matrix} \right\} \left\{ \begin{matrix} \text{Ou} \\ \text{Re} \end{matrix} \right\} X_{qq'}^1 = \begin{cases} 0 & \text{for OuOu or ReRe.} \\ i\delta_{qq'} & \begin{cases} (+1) & \text{for OuRe.} \\ (-1) & \text{for ReOu.} \end{cases} \end{cases} \quad (3.18)$$

Expressions similar to (3.18) has been studied for the acoustic and electromagnetic cases in (21) and (22), respectively. In (7)  $X^1$  is expressed as the difference between the Q-matrices for a cavity and a rigid body and the result (3.18) is obtained there. More interesting is perhaps the possibility to develop matrix formulations without using Green's functions or dyadics. This is done by constructing, for each kind of problem (acoustic, electromagnetic or elastic), the quantity corresponding to the integrand in (3.17), see (8). Using the remark about the T-matrix, just after equation (3.5), and the relation (3.18) we get (3.19) from (3.15) and (3.16).

$$\vec{\beta}^1 = T^2 \vec{\alpha}^1 \quad (3.19)$$

Here  $T^2$  is the T-matrix for the body inside the surface  $S_2$  (i.e. the inclusion). We notice that the material parameters inside the surface  $S_2$  can be dependent on the space coordinates. Furthermore, the surface  $S_2$  might be interpreted solely as a mathematical abstraction (i.e., it doesn't have to be the boundary between different materials). Except for possible separated inclusions within  $S_1$  the material contained in  $S_1$  and  $S_2$  may be identical. From (3.12), (3.13) and (3.19) we derive the T-matrix for the entire layered elastic body as:

$$T = - [ \text{ReRe} Q^1 + \text{ReOu} Q^1 T^2 ] [ \text{OuRe} Q^1 + \text{OuOu} Q^1 T^2 ]^{-1}. \quad (3.20)$$

We will end this section by deriving the field between the surfaces  $S_1$  and  $S_2$  in the body. By using the integral representation (2.8) and considering  $\vec{r}$  in the region between the sphere  $Z_1$  and the sphere  $X_2$ , applying the expansions (3.8) and (2.23), we get:

$$\begin{aligned} \vec{u}_1 &= -i \sum_{q,q'} \left( \text{OuRe} X_{qq'}^1, \text{Re} \vec{\psi}_q^{+1} \alpha_q^1 + \text{OuOu} X_{qq'}^1, \text{Re} \vec{\psi}_q^{+1} \beta_q^1 \right) + \\ &\quad + \vec{q} \left( i \frac{\kappa_1^3}{\rho_1 \omega^2} \int_{S_2} [ \vec{t}_1 (\text{Re} \vec{\psi}_q^{+1}) \cdot \vec{u}_{1+} - \text{Re} \vec{\psi}_q^{+1} \cdot \vec{t}_{1+} (\vec{u}_1) ] ds \right) \vec{\psi}_q^{+1} \end{aligned} \quad (3.21)$$

Using the remark about the T-matrix, just after equation (3.5), and the relations (3.18) and (3.19), we get (3.22) from (3.21).

$$\vec{u}_1 = \sum_q \left( \alpha_q^1 \text{Re} \vec{\psi}_q^{+1} + \beta_q^1 \vec{\psi}_q^{+1} \right) \quad (3.22)$$



This relation is of course only valid when the sphere  $X_2$  is truly inside the sphere  $Z_1$ .

#### IV. LAYERED BODIES WITH AN ELASTIC MATERIAL INSIDE THE BOUNDING SURFACE AND A FLUID OUTSIDE

Consider an elastic body, with an inclusion, situated in an infinite fluid. Outside the bounding surface  $S_1$  as well as between  $S_1$  and an inner surface  $S_2$  the material parameters are constant and given in the notations from section II. (notice that  $u_0 = 0$ ). The main difference from section III is that in this section we only have to deal with the curl free basis functions outside  $S_1$  (in the fluid). This, in turn, will lead to some minor complications. The expansions of the incoming and scattered field (in the fluid) is given by (4.1) and (4.2), respectively.

$$\vec{u}_0^i(\vec{r}) = \sum_p a_p \text{Re} \vec{\psi}_{3p}^0(\vec{r}) \quad \text{for } r < r^{so} \quad (4.1)$$

$$\vec{u}_0^s(\vec{r}) = \sum_p f_p \vec{\psi}_{3p}^0(\vec{r}) \quad \text{for } r > r_{\max}^1 \quad (4.2)$$

Here,  $r^{so}$  and  $r_{\max}^1$  has the same meaning as in section III. By a procedure similar to the one used in section III we get (4.3) and (4.4). Notice that we have  $k_0$  here instead of  $\kappa_0$  as in section III.

$$f_p = i \frac{k_0^3}{\rho_0 \omega^2} \int_{S_1} [\vec{t}_0(\text{Re} \vec{\psi}_{3p}^0) \cdot \vec{u}_{0+} - \text{Re} \vec{\psi}_{3p}^0 \cdot \vec{t}_{0+}(\vec{u}_0)] ds \quad (4.3)$$

$$a_p = -i \frac{k_0^3}{\rho_0 \omega^2} \int_{S_1} [\vec{t}_0(\vec{\psi}_{3p}^0) \cdot \vec{u}_{0+} - \vec{\psi}_{3p}^0 \cdot \vec{t}_{0+}(\vec{u}_0)] ds \quad (4.4)$$

The total T-matrix for the layered body is defined directly by (4.5) and indirectly by (4.3) and (4.4).

$$f_p = \sum_p T_{pp} a_p \quad (4.5)$$

The boundary conditions at the surface  $S_1$  is given by (4.6), (4.7) and (4.8)

$$\hat{n}_1 \cdot \vec{u}_{0+} = \hat{n}_1 \cdot \vec{u}_{1-} \quad (\text{for } \vec{r} \in S_1) \quad (4.6)$$

$$\hat{n}_1 \cdot \vec{t}_{0+} = \hat{n}_1 \cdot \vec{t}_{1-} \quad (\text{for } \vec{r} \in S_1) \quad (4.7)$$

$$[\vec{t}_{0+}]_{\tan} = [\vec{t}_{1-}]_{\tan} = 0 \quad (\text{for } \vec{r} \in S_1) \quad (4.8)$$

Here,  $[\ ]_{\tan}$  stands for the tangential component. The tangential components of  $\vec{u}_{0+}$  and  $\vec{u}_{1-}$  are unrelated. However, because of the relation (4.9) we notice that no information about the tangential part of  $\vec{u}_{0+}$  or  $\vec{u}_{1-}$  is needed.

$$\vec{t}_0(\vec{v}_0) = \hat{n}_1 \cdot \vec{\sigma}_0(\vec{v}_0) = \lambda_0 (\nabla \cdot \vec{v}_0) \hat{n}_1 = (\hat{n}_1 \cdot \vec{t}_0(\vec{v}_0)) \hat{n}_1 \quad (4.9)$$



Using (4.6), (4.7), (4.9) and the expansion (3.8) in (4.3) and (4.4), we get:

$$\vec{f}_p = i\vec{\gamma}_q, (\text{ReRe}Q_{pq}, \alpha_q^1, + \text{ReOu}Q_{pq}, \beta_q^1, ) \quad (4.10)$$

$$\vec{a}_p = -i\vec{\gamma}_q, (\text{OuRe}Q_{pq}, \alpha_q^1, + \text{OuOu}Q_{pq}, \beta_q^1, ) , \quad (4.11)$$

where the matrix Q is given by

$$\begin{aligned} \left\{ \begin{matrix} \text{Ou} \\ \text{Re} \end{matrix} \right\} \left\{ \begin{matrix} \text{Ou} \\ \text{Re} \end{matrix} \right\} Q_{pq} = & \frac{k_0^3}{\rho_0 \omega^2} \int_{S_1} \left[ \lambda_0 \vec{\nabla} \cdot \left\{ \begin{matrix} \text{Ou} \\ \text{Re} \end{matrix} \right\} \vec{\psi}_{3p}^0 \hat{n}_1 \cdot \left\{ \begin{matrix} \text{Ou} \\ \text{Re} \end{matrix} \right\} \vec{\psi}_q^1, \right. \\ & \left. - \hat{n}_1 \cdot \left\{ \begin{matrix} \text{Ou} \\ \text{Re} \end{matrix} \right\} \vec{\psi}_{3p}^0 \hat{n}_1 \cdot \vec{t}_1 \left( \left\{ \begin{matrix} \text{Ou} \\ \text{Re} \end{matrix} \right\} \vec{\psi}_q^1 \right) \right] ds . \end{aligned} \quad (4.12)$$

In vector matrix notations we rewrite (4.10) and (4.11) as:

$$\vec{f} = i\text{ReRe}Q\vec{\alpha}^1 + i\text{ReOu}Q\vec{\beta}^1 \quad (4.13)$$

$$\vec{a} = -i\text{OuRe}Q\vec{\alpha}^1 - i\text{OuOu}Q\vec{\beta}^1 . \quad (4.14)$$

Notice that the indices in the matrix Q belong to different sets (i.e., it can not be made square by choosing different finite truncations in the two indices n and n'. The most obvious counterexample is a sphere.) This means that even if the body under consideration is homogeneous (no inclusion) which, in turn, implies that  $\vec{\beta}^1 \equiv 0$  we can not yet solve the equations (4.13) and (4.14) by elimination of  $\vec{\alpha}^1$ . To overcome this we introduce the expansion (4.15) of the field on the outside of  $S_1$ .

$$\vec{u}_{0+} = \sum_p \gamma_p \text{Re} \vec{\psi}_{3p}^0 \quad (4.15)$$

By using the integral representation (2.8) and considering  $\vec{r}$  outside the sphere  $X_1$ , applying the expansion (4.15), and applying the boundary conditions (4.6), (4.7), and (4.8), we get (4.16), after identifying coefficients.

$$\begin{aligned} \sum_p \gamma_p \text{ReRe}P_{qp}, \gamma_p, + \sum_q, (\text{ReRe}R_{qq}, \alpha_q^1, + \text{ReOu}R_{qq}, \beta_q^1, ) \\ - \frac{\kappa_1^3}{\rho_1 \omega^2} \int_{S_2} [\vec{t}_1 (\text{Re} \vec{\psi}_q^1) \vec{u}_{1+} - \text{Re} \vec{\psi}_q^1 \cdot \vec{t}_{1+} (\vec{u}_1)] ds = 0 \end{aligned} \quad (4.16)$$

where

$$\left\{ \begin{matrix} \text{Ou} \\ \text{Re} \end{matrix} \right\} \left\{ \begin{matrix} \text{Ou} \\ \text{Re} \end{matrix} \right\} P_{qp} = \frac{\kappa_1^3}{\rho_1 \omega^2} \int_{S_1} \vec{t}_1 \left( \left\{ \begin{matrix} \text{Ou} \\ \text{Re} \end{matrix} \right\} \vec{\psi}_q^1 \right) \cdot \hat{n}_1 \hat{n}_1 \cdot \left\{ \begin{matrix} \text{Ou} \\ \text{Re} \end{matrix} \right\} \vec{\psi}_{3p}^0, ds \quad (4.17)$$

and

$$\begin{aligned} \left\{ \begin{matrix} \text{Ou} \\ \text{Re} \end{matrix} \right\} \left\{ \begin{matrix} \text{Ou} \\ \text{Re} \end{matrix} \right\}_{R_{qq'}} &= \frac{\kappa_1^3}{\rho_1 \omega^2} \int_{S_1} \left[ \vec{t}_1 \left( \left\{ \begin{matrix} \text{Ou} \\ \text{Re} \end{matrix} \right\}_{\vec{\psi}_q}^1 \right) \cdot \left[ \left\{ \begin{matrix} \text{Ou} \\ \text{Re} \end{matrix} \right\}_{\vec{\psi}_{q'}}^1 \right] \tan \right. \\ &\quad \left. - \left\{ \begin{matrix} \text{Ou} \\ \text{Re} \end{matrix} \right\}_{\vec{\psi}_q}^1 \cdot \hat{n}_1 \hat{n}_1 \cdot \vec{t}_1 \left( \left\{ \begin{matrix} \text{Ou} \\ \text{Re} \end{matrix} \right\}_{\vec{\psi}_{q'}}^1 \right) \right] ds. \end{aligned} \quad (4.18)$$

We rewrite (4.16) in a vector matrix notation using (3.19) and the remark about the T-matrix, just after equation (3.5).

$$\text{ReReP}\vec{\gamma} + \text{ReReR}\vec{\alpha}^1 + \text{ReOuRT}^2\vec{\alpha}^1 + i\text{T}\vec{\alpha}^1 = 0 \quad (4.19)$$

We also rewrite (4.13) and (4.14) after using (3.19).

$$\vec{f} = i\text{ReReQ}\vec{\alpha}^1 + i\text{ReOuQT}^2\vec{\alpha}^1 \quad (4.20)$$

$$\vec{a} = -i\text{OuReQ}\vec{\alpha}^1 - i\text{OuOuQT}^2\vec{\alpha}^1 \quad (4.21)$$

If the body under consideration is homogeneous (no inclusion) we have  $\text{T}^2 \equiv 0$ . The T-matrix for the homogeneous body with no inclusion is given by (4.22).

$$\text{T} = -\text{ReReQ}(\text{ReReR})^{-1}\text{ReReP}[\text{OuReQ}(\text{ReReR})^{-1}\text{ReReP}]^{-1} \quad (4.22)$$

The T-matrix for the entire layered elastic body with an inclusion possibly as general as described in section III, is given by (4.23).

$$\begin{aligned} \text{T} = & -(\text{ReReQ} + \text{ReOuQT}^2)(\text{ReReR} + \text{ReOuRT}^2 + i\text{T}^2)^{-1}\text{ReReP} \times \\ & \times [(\text{OuReQ} + \text{OuOuQT}^2)(\text{ReReR} + \text{ReOuRT}^2 + i\text{T}^2)^{-1}\text{ReReP}]^{-1} \end{aligned} \quad (4.23)$$

#### V. LAYERED BODIES WITH A FLUID INSIDE THE BOUNDING SURFACE AND AN ELASTIC MATERIAL OUTSIDE

Consider a bounded volume of fluid, with an inclusion, situated in an infinite elastic material. Outside the bounding surface  $S_1$  as well as between  $S_1$  and an inner surface  $S_2$  the material parameters are constant and given in the notations from section II (notice that  $\mu_1 = 0$ ). We can proceed as in section III using the integral representation (2.7), the expansions (2.23), (3.1), and (3.2) to obtain the formulas (3.3) and (3.4), which are applicable here also. The boundary conditions are given by (4.6), (4.7), and (4.8) in section IV. We now introduce the expansions (5.1) and (5.2) for the field on the outside of  $S_1$  and on the inside of  $S_1$ , respectively.

$$\vec{u}_{0+} = \sum_q \gamma_q \text{Re}\vec{\psi}_q^0 \quad (5.1)$$

$$\vec{u}_{1-} = \sum_p (\alpha_p^1 \text{Re}\vec{\psi}_{3p}^1 + \beta_p^1 \vec{\psi}_{3p}^1) \quad (5.2)$$

Notice, that outside  $S_1$  (in the elastic material) we need both curl and divergence free basis functions. But, inside  $S_1$  (in the fluid) we need only the curl free basis functions. Using the boundary conditions (4.6), (4.7), and (4.8) together with the expansions (5.1) and (5.2) in (3.3) and (3.4), we get:

$$\vec{f} = i\text{ReReU}\vec{\gamma} + i\text{ReReV}\vec{\alpha}^1 + i\text{ReOuV}\vec{\beta}^1 \quad (5.3)$$

$$\vec{a} = -i\text{OuReU}\vec{\gamma} - i\text{OuReV}\vec{\alpha}^1 - i\text{OuOuV}\vec{\beta}^1, \quad (5.4)$$

where the matrices U and V are given by:

$$\left\{ \begin{matrix} \text{Ou} \\ \text{Re} \end{matrix} \right\}_{\text{ReU}}{}_{\text{qq}} = \frac{\kappa_0^3}{\rho_0 \omega^2} \int_{S_1} \vec{t}_0 \left( \left\{ \begin{matrix} \text{Ou} \\ \text{Re} \end{matrix} \right\}_{\vec{\psi}_q}^0 \right) \cdot \left[ \text{Re}\vec{\psi}_q^0 \right] \tan ds \quad (5.5)$$

$$\begin{aligned} \left\{ \begin{matrix} \text{Ou} \\ \text{Re} \end{matrix} \right\}_{\text{Re}} \left\{ \begin{matrix} \text{Ou} \\ \text{Re} \end{matrix} \right\}_{\text{V}}{}_{\text{qp}} &= \frac{\kappa_0^3}{\rho_0 \omega^2} \int_{S_1} \left[ \vec{t}_0 \left( \left\{ \begin{matrix} \text{Ou} \\ \text{Re} \end{matrix} \right\}_{\vec{\psi}_q}^0 \right) \cdot \hat{n}_1 \hat{n}_1 \cdot \left\{ \begin{matrix} \text{Ou} \\ \text{Re} \end{matrix} \right\}_{\vec{\psi}_{3p}}^1 \right. \\ &\quad \left. - \left\{ \begin{matrix} \text{Ou} \\ \text{Re} \end{matrix} \right\}_{\vec{\psi}_q}^0 \cdot \hat{n}_1 \lambda_1 \nabla \cdot \left\{ \begin{matrix} \text{Ou} \\ \text{Re} \end{matrix} \right\}_{\vec{\psi}_{3p}}^1 \right] ds. \end{aligned} \quad (5.6)$$

Notice that U is "square" but V is not. Here as in section IV we can see that even if the fluid is homogeneous (no inclusions) which, in turn, implies that  $\vec{B}^1 \equiv 0$  we can not yet solve the equations (5.3) and (5.4). We proceed in a way similar to section III with two exceptions. First, unlike section III where  $\kappa_1$  was used, we shall substitute  $k_1$  in the integral representation (2.8). Second, unlike section III where both curl and divergence free basis functions were used we shall expand the various fields between  $S_1$  and  $S_2$  using curl free basis functions alone. By using the integral representation (2.8) (with  $k_1$  instead of  $\kappa_1$ ) and considering  $\vec{r}$  outside the sphere  $X_1$ , applying the expansions (5.2), we get (5.7), after identifying coefficients. Similarly, considering  $\vec{r}$  inside the sphere  $Z_2$  we get (5.8).

$$\begin{aligned} \Sigma_p, (\text{ReReX}_{3p3p}^1, \alpha_p^1, + \text{ReOuX}_{3p3p}^1, \beta_p^1, ) \\ - \frac{k_1^3}{\rho_1 \omega^2} \int_{S_2} [\vec{t}_1 (\text{Re}\vec{\psi}_{3p}^1) \cdot \vec{u}_{1+} - \text{Re}\vec{\psi}_{3p}^1 \cdot \vec{t}_{1+} (\vec{u}_1)] ds = 0 \end{aligned} \quad (5.7)$$

$$\begin{aligned} \Sigma_p, (\text{OuReX}_{3p3p}^1, \alpha_p^1, + \text{OuOuX}_{3p3p}^1, \beta_p^1, ) \\ - \frac{k_1^3}{\rho_1 \omega^2} \int_{S_2} [\vec{t}_1 (\vec{\psi}_{3p}^1) \cdot \vec{u}_{1+} - \vec{\psi}_{3p}^1 \cdot \vec{t}_{1+} (\vec{u}_1)] ds = 0 \end{aligned} \quad (5.8)$$

Notice that two modifications have been made when changing from an elastic medium as in section III to a fluid. The first is the change from  $\kappa_1$  to  $k_1$  in the integral representation (2.8). The second modification is the removal of the factor  $(k_1/\kappa_1)^{3/2}$  in the curl free basis function. These two changes cancel each other out. For this reason we obtain a subpart of the matrix  $X^1$  as defined in (3.17). By using the properties of the matrix  $X^1$  given by (3.18) and the remark about the T-matrix after equation (3.5), we get:

$$\vec{\beta}^1 = T^2 \vec{\alpha}^1, \quad (5.9)$$



where  $T^2$  is the T-matrix for the body inside the surface  $S_2$  (i.e. the inclusion). The equations (5.3), (5.4), and (5.9) are still not sufficient to give us the T-matrix for the total body. We have to use the integral representation (2.8), but now with the expansion (5.1) introduced through the boundary conditions at  $S_1$ . By using the integral representation (2.8) (with  $k_1$  instead of  $\kappa_1$ ) and considering  $\vec{r}$  outside the sphere  $X_1$ , applying the expansions (5.1) and (5.2), and applying the boundary conditions (4.6) we get (5.10).

$$\begin{aligned} \sum_q \text{ReReW}_{3pq} \gamma_q - \sum_p (\text{ReReY}_{3p3p} \alpha_p^1 + \text{ReOuY}_{3p3p} \beta_p^1) \\ - \frac{k_1^3}{\rho_1 \omega^2} \int_{S_2} [\vec{t}_1 (\text{Re}\vec{\psi}_{3p}^1) \cdot \vec{u}_{1+} - \text{Re}\vec{\psi}_{3p}^1 \cdot \vec{t}_{1+} (\vec{u}_1)] ds = 0 \end{aligned} \quad (5.10)$$

where

$$\left\{ \begin{matrix} \text{Ou} \\ \text{Re} \end{matrix} \right\} \text{ReW}_{3pq} = \frac{k_1^3}{\rho_1 \omega^2} \int_{S_1} \lambda_1 \nabla \cdot \left\{ \begin{matrix} \text{Ou} \\ \text{Re} \end{matrix} \right\} \vec{\psi}_{3p}^1 \hat{n}_1 \cdot \text{Re}\vec{\psi}_q^0 ds \quad (5.11)$$

$$\left\{ \begin{matrix} \text{Ou} \\ \text{Re} \end{matrix} \right\} \left\{ \begin{matrix} \text{Ou} \\ \text{Re} \end{matrix} \right\} \text{Y}_{3p3p} = \frac{k_1^3}{\rho_1 \omega^2} \int_{S_1} \left\{ \begin{matrix} \text{Ou} \\ \text{Re} \end{matrix} \right\} \vec{\psi}_{3p}^1 \cdot \hat{n}_1 \lambda_1 \nabla \cdot \left\{ \begin{matrix} \text{Ou} \\ \text{Re} \end{matrix} \right\} \vec{\psi}_{3p}^1 ds \quad (5.12)$$

Using (3.19) and the remark about the T-matrix, just after equation (3.5) we get (5.13) from (5.10).

$$\text{ReReW}^{\vec{a}} - \text{ReReY}^{\vec{a}1} - \text{ReOuYT}^{\vec{a}1} + iT^{\vec{a}1} = 0 \quad (5.13)$$

We also rewrite (5.3) and (5.4) after using (3.19).

$$\vec{f} = i\text{ReReU}^{\vec{a}} + i\text{ReReV}^{\vec{a}1} + i\text{ReOuVT}^{\vec{a}1} \quad (5.14)$$

$$\vec{a} = -i\text{OuReU}^{\vec{a}} - i\text{OuReV}^{\vec{a}1} - i\text{OuOuVT}^{\vec{a}1} \quad (5.15)$$

If the body under consideration is homogeneous (no inclusion) we have  $T^2 \equiv 0$ . The T-matrix for the homogeneous "fluid-body" with no inclusion is given by (5.16).

$$\begin{aligned} T = - (\text{ReReU} + \text{ReReV}(\text{ReReY})^{-1} \text{ReReW})(\text{OuReU} \\ + \text{OuReV}(\text{ReReY})^{-1} \text{ReReW})^{-1} \end{aligned} \quad (5.16)$$

The T-matrix for the entire layered "fluid-body", with an inclusion possibly as general as described in section III, is given by (5.17).

$$\begin{aligned} T = - [\text{ReReU} + (\text{ReReV} + \text{ReOuVT}^2)(\text{ReReY} + \text{ReOuYT}^2 - iT^2)^{-1} \text{ReReW}] \times \\ \times [\text{OuReU} + (\text{OuReV} + \text{OuOuVT}^2)(\text{ReReY} + \text{ReOuYT}^2 - iT^2)^{-1} \text{ReReW}]^{-1} \end{aligned} \quad (5.17)$$



## VI. MULTIPLE SCATTERING

Consider  $N$  elastic bodies situated in an infinite elastic material. Outside the bounding surfaces of the  $N$  bodies the material parameters are constant and given in the notations from section II. The integral representation (2.7) can be used after changing the region of integration. We then have the following expression:

$$\begin{aligned} \vec{u}_0^i(\vec{r}) + \frac{\kappa_0^3}{\rho_0 \omega^2} \int_{S_i} [\vec{u}_{0+} \cdot (\vec{n}_i \cdot \vec{e}_0) - \vec{n}_i \cdot \vec{\sigma}_{0+} \cdot \vec{e}_0] ds' = \\ = \begin{cases} \vec{u}_0(\vec{r}), & \vec{r} \text{ outside } S_i \text{ for all } i \leq N. \\ 0, & \vec{r} \text{ inside } S_i \text{ for a specific } i \leq N. \end{cases} \end{aligned} \quad (6.1)$$

For the expansion of the Green's dyadic and the incoming field we use (2.23) and (3.1), respectively. However, for the scattered field we now must deal with two different regions. One region is the volume outside of the smallest sphere circumscribing all the surfaces  $S_i$  (see Fig. 2). The other region is the volume inside the largest sphere inscribed within all the surfaces  $S_i$ . Both spheres have center at the origin  $O$ . The radii of these spheres are called  $r^{\text{out}}$  and  $r^{\text{reg}}$ , respectively (see Fig. 2). We now expand the scattered fields as follows:

$$\vec{u}_0^s = \sum_q f_q^{\text{reg}} \text{Re} \vec{\psi}_q^0(\vec{r}) \quad \text{for } r < r^{\text{reg}} \quad (6.2)$$

$$\vec{u}_0^s = \sum_q f_q^{\text{out}} \vec{\psi}_q^0(\vec{r}) \quad \text{for } r > r^{\text{out}} \quad (6.3)$$

Before proceeding further we require the translation properties for the basis functions. These are given by the following formulas:

$$\vec{\psi}_q^i(\vec{r} + \vec{a}) = \begin{cases} \sum_q R(\vec{a})_{qq} \vec{\psi}_q^i(\vec{r}) & \text{for } a < r. \\ \sum_q \sigma(\vec{a})_{qq} \text{Re} \vec{\psi}_q^i(\vec{r}) & \text{for } a > r, \end{cases} \quad (6.4)$$

$$\text{Re} \vec{\psi}_q^i(\vec{r} + \vec{a}) = \sum_q R(\vec{a})_{qq} \text{Re} \vec{\psi}_q^i(\vec{r}) \quad \text{for all } a \text{ and } r. \quad (6.5)$$

The properties of the matrices  $R$  and  $\sigma$  for the acoustic and electromagnetic problems are treated in (24) and (23). Here we simply need the exterior product of these two kinds of representations. The idea is to express the surface integrals in terms of the radius vectors  $\vec{r}_i$  extending from the origins  $O_i$ . These coordinate systems are pure translations of each other (see Fig. 2). By considering  $\vec{r}$  inside the inscribed sphere of surface  $S_i$ , with center at the origin  $O_i$ , we obtain (6.6) by using the expansions (3.1), (2.23), (6.4), and (6.5) and by identifying the coefficients of the regular functions.

$$R^t(\vec{a}_i) \vec{a} = \vec{a}^i - \sum_{j \neq i} \sigma(-\vec{a}_i + \vec{a}_j) \vec{f}^j \quad \text{for } i, j = 1, 2, 3 \dots N, \quad (6.6)$$

with the restriction:

$$|-\vec{a}_i + \vec{a}_j| > r_i', r_j' . \quad (6.7)$$

The superscript  $t$  in (6.6) stands for the transpose of  $R$  (which in turn is equal to the inverse of  $R$ ). The double prime on  $\vec{r}_i$  in (6.7) indicates that  $\vec{r}_i$  is on the surface  $S_i$ . Here we have used the following expressions for the components of the vectors  $\vec{a}^i$  and  $\vec{f}^i$ .

$$\vec{f}_q^i = i \frac{k_0^3}{\rho_0 \omega^2} \int_{S_i} [\vec{t}_0(\text{Re} \vec{\psi}_q^0(\vec{r}_i')) \cdot \vec{u}_{0+} - \text{Re} \vec{\psi}_q^0(\vec{r}_i') \cdot \vec{t}_{0+}(\vec{u}_0)] ds' \quad (6.8)$$

$$\vec{a}_q^i = -i \frac{k_0^3}{\rho_0 \omega^2} \int_{S_i} [\vec{t}_0(\vec{\psi}_q^0(\vec{r}_i')) \cdot \vec{u}_{0+} - \vec{\psi}_q^0(\vec{r}_i') \cdot \vec{t}_{0+}(\vec{u}_0)] ds' \quad (6.9)$$

By considering  $\vec{r}$  outside the sphere, with radius  $r^{\text{out}}$ , mentioned above, we obtain (6.10) by using the expansions (6.3), (2.23), (6.5) and by identifying coefficients of the outgoing functions.

$$\vec{f}^{\text{out}} = \sum_i R(\vec{a}_i) \vec{f}^i \quad (6.10)$$

Further, by considering  $\vec{r}$  inside the sphere with radius  $r^{\text{reg}}$ , mentioned above, we obtain (6.11) by using the expansions (6.2), (2.23), (6.4) and by identifying coefficients of the regular functions.

$$\vec{f}^{\text{reg}} = \sum_i \sigma(\vec{a}_i) \vec{f}^i \quad (6.11)$$

We observe that the T-matrix  $T_i$  for the body bounded by the surface  $S_i$  gives the relation between  $\vec{f}^i$  and  $\vec{a}^i$  as follows:

$$\vec{f}^i = T_i \vec{a}^i . \quad (6.12)$$

We thus obtain the following system of algebraic equations with unknown coefficients  $\vec{a}^i$ , which can be eliminated:

$$R^t(\vec{a}_i) \vec{a}^i = \vec{a}^i - \sum_{j \neq i} \sigma(-\vec{a}_i + \vec{a}_j) T_j \vec{a}^j \quad \text{for } i, j = 1, 2, 3, \dots, N , \quad (6.13)$$

with the restriction:  $|-\vec{a}_i + \vec{a}_j| > r_i' ,$

$$\vec{f}^{\text{out}} = \sum_i R(\vec{a}_i) T_i \vec{a}^i , \quad (6.14)$$

$$\vec{f}^{\text{reg}} = \sum_i \sigma(\vec{a}_i) T_i \vec{a}^i . \quad (6.15)$$

The separation requirement is weaker than separability by planes or nonoverlapping spheres. We note that the result (6.13)-(6.15) is independent of whether the infinite medium is a fluid (with  $\mu_0 = 0$ ) or a fully elastic material (with  $\mu_0 \neq 0$ ). The differences in the derivation when the infinite material is a fluid is that  $k_0$  has to be changed to  $k_0$  and only the curl free basis function is needed (i.e.  $\tau = 3$ ). Furthermore, it is clear the kind of material inside  $S_i$  is irrelevant as

pointed out before. Equation (6.16) gives the T-matrix for all the N bodies as a formal solution of (6.13) and (6.14).

$$T = \sum_{i,j} R(\vec{a}_i) T_i \{ \delta_{mn} - \sum_{n \neq m} \sigma(-\vec{a}_m + \vec{a}_n) T_n \}^{-1} R^t(\vec{a}_j) \quad (6.16)$$

Here, the symbol  $\{ \}$  stands for a matrix of the space point indices. The elements of this matrix don't commute which makes the inversion more difficult than in the case of matrices with elements which are complex numbers. A procedure to obtain the T-matrix in a maximally symmetric form is described in (23) and (24). The T-matrix for two and three scatterers is also obtained and interpreted in physical terms in these references.

## VII. EXTENSIONS

Let us consider a body with N consecutively enclosing surfaces  $S_i$  ( $i = 1, 2, \dots, N$ ) separating different elastic materials (see Fig. 3). We associate the Q-matrix  $Q^i$ , given by (3.11), with surface  $S_i$ . The T-matrix  $T^i$  is associated with the body inside the surface  $S_i$ . It is clear that the formula (3.21) can be generalized to (7.1) relating  $T^{i-1}$  with  $T^i$  by means of  $Q^{i-1}$ .

$$T^{i-1} = - [ReReQ^{i-1} + ReOuQ^{i-1} T^i] [OuReQ^{i-1} + OuOuQ^{i-1} T^i]^{-1} \quad (7.1)$$

Starting with the T-matrix  $T^N$  for the innermost surface  $S_N$  and the Q-matrix  $Q^{N-1}$  for the next innermost surface  $S_{N-1}$  we get, using (7.1), the T-matrix  $T^{N-1}$  for the body inside the surface  $S_{N-1}$ . Repeated use of (7.1) finally gives us the T-matrix  $T^1$  for the entire body.

As mentioned before in section III the innermost surface  $S_N$  may be interpreted solely as a mathematical abstraction and, hence, may have no physical interpretation. The surface  $S_N$  may contain a set of nonintersecting, nonembedded surfaces  $S^i$  ( $i = 1, 2, \dots, M$ ). These surfaces ( $S^i$ ), in turn, may enclose separate regions having different material properties (see Fig. 4). In this case we get the T-matrix  $T^N$  for the M bodies inside  $S_N$  by the methods in section VI.

The body with the N surfaces  $S_i$ , with which we started this section, can be considered together with other neighboring bodies. These neighboring bodies can be enclosed by different layers of material. Again, the total T-matrix can be computed as outlined above.

We can generalize equation (7.1) producing:

$$T^{i-1} = f(Q^{i-1}, T^i) \quad (7.2)$$

Equation (4.23) for the body in a fluid, and equation (5.17) for the "fluid-body" in an elastic material can be generalized to (7.3) and (7.4), respectively.

$$T^{i-1} = g(P^{i-1}, Q^{i-1}, R^{i-1}, T^i) \quad (7.3)$$

$$T^{i-1} = h(U^{i-1}, V^{i-1}, W^{i-1}, Y^{i-1}, T^i) \quad (7.4)$$

The structure of equations (7.2)-(7.4) implies we can relax our initial restriction to elastic inclusions. Starting from some innermost surface and using either equation (7.2), (7.3), (7.4), or (6.16) we can get the T-matrix for most combinations of fluid and elastic regions. Consider, for example, the three regions in



Fig. 5 where  $S_{01}$ ,  $S_{02}$  and  $S_{12}$  are separating different materials. This configuration can, as shown in (21) and (22), be treated as multiple scattering from two bodies. However, the origins of the various coordinate systems involved must be chosen so that the geometrical constraint in (6.13) is satisfied. The radius vector describing the surfaces also must be single valued.

We can also study multiple scattering in lattices, with unit cells composed of more than one scatterer, as in (26). In this case we need only plug in the proper matrices pertinent to our elastic-fluid problem in the formulas, as given in (26).

### VIII. CROSS SECTIONS

In this section we begin with wave scattering in fluids. As mentioned earlier this can be described by a scalar potential. The expansion  $\phi_0^s(\vec{r}) = \sum_p f_p^s \phi_p^0(\vec{r})$  ( $\phi_p^0$  is given by (2.22)) of the scattered scalar field  $\phi_0^s$  gives us the following asymptotic behavior of the scattered field:

$$\phi_0^s(\vec{r}) = \frac{1}{k_0} \sum_p (-1)^{n+1} f_p^s Y_p(\theta, \phi) \frac{e^{ik_0 r}}{r} \quad \text{for large } r. \quad (8.1)$$

This leads us to the following definition of the scattering amplitude  $A^s$ , which is independent of the radius  $r$ :

$$A^s(\theta, \phi) = \lim_{r \rightarrow \infty} r e^{-ik_0 r} \phi_0^s(\vec{r}). \quad (8.2)$$

We are going to define the various cross sections (32,33,34) in terms of incoming plane waves. For this case the cross sections can be interpreted physically. This physical interpretation will be treated later.

The incoming plane wave  $\phi_0^i$  is given by:

$$\phi_0^i(\vec{r}) = C e^{i\vec{k}_0 \cdot \vec{r}} = \sum_p a_p^{pl}(\alpha, \beta) \text{Re} \phi_p^0(\vec{r}) = 4\pi C \sum_p i^n Y_p(\alpha, \beta) \text{Re} \phi_p^0(\vec{r}). \quad (8.3)$$

Here, the angles  $\alpha$  and  $\beta$  are the spherical angles for the incoming wave vector  $\vec{k}_0$  (see Fig. 6). This wave vector is given by:

$$\vec{k}_0 = k_0(\sin\alpha \cos\beta, \sin\alpha \sin\beta, \cos\alpha). \quad (8.4)$$

We define the differential scattering cross section  $d\sigma/d\Omega$  by:

$$\frac{d\sigma(\alpha, \beta, \theta, \phi)}{d\Omega(\theta, \phi)} = \lim_{r \rightarrow \infty} r^2 \left| \frac{\phi_0^s}{\phi_0^i} \right|^2 = \left| \frac{A^s(\alpha, \beta, \theta, \phi)}{C} \right|^2, \quad (8.5)$$

where  $d\Omega(\theta, \phi) = \sin\theta d\theta d\phi$ . Again,  $\alpha$  and  $\beta$  are the spherical angles for the incoming wave. Observe that both the scattered field  $\phi_0^s$  and the scattering amplitude  $A^s$  are functions of these angles, as shown in equation (8.5). An incoming plane wave, with direction defined by  $\alpha$  and  $\beta$ , produces wave scattering in the direction given by  $\theta$  and  $\phi$ . The amount of energy scattered is proportional to the differential scattering cross section  $d\sigma/d\Omega$ .



The total scattering cross section  $\sigma$  is defined by:

$$\sigma(\alpha, \beta) \equiv \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{d\sigma}{d\Omega} d\Omega = \frac{1}{|C|^2 k_0^2} \sum_p |f_p^s(\alpha, \beta)|^2. \quad (8.6)$$

Here, the coefficients  $f_p^s$  in the expansion of the scattered field depend on the spherical angles  $\alpha$  and  $\beta$ , which, in turn, define the direction of the incoming plane wave. The total scattering cross section is proportional to the total amount of energy scattered in all directions.

The extinction (or total) cross section  $\sigma_e$  is defined by:

$$\sigma_e(\alpha, \beta) \equiv \frac{4\pi}{k_0} \text{Im} \frac{1}{C} A^s(\alpha, \beta, \alpha, \beta). \quad (8.7)$$

Here,  $\text{Im}$  stands for "the imaginary part of", and  $C$  is the amplitude of the incoming plane wave. The extinction cross section is proportional to the total amount of energy lost from an incoming plane wave with direction given by  $\alpha$  and  $\beta$ .

Finally, we define the absorption cross section  $\sigma_a$  by:

$$\sigma_a(\alpha, \beta) \equiv \sigma_e(\alpha, \beta) - \sigma(\alpha, \beta) \quad (8.8)$$

This cross section is proportional to the amount of energy that is absorbed by the body from the incoming plane wave. The angles  $\alpha$  and  $\beta$  give the direction of this incoming wave.

Instead of using the potential to describe the scattering in a fluid we can use the displacement field. The expansion  $\vec{u}_0^s(\vec{r}) = \sum_p f_p \vec{\psi}_{3p}^0(\vec{r})$  of the scattered displacement field  $\vec{u}_0^s$  gives us the following asymptotic behavior of the scattered field:

$$\vec{u}_0^s(\vec{r}) = \frac{\hat{r}}{k_0} \sum_p (-1)^n f_p Y_p(\theta, \phi) \frac{e^{ik_0 r}}{r} \quad \text{for large } r. \quad (8.9)$$

We define the scattering amplitude  $\vec{A}$ , which now is a vector in the  $\hat{r}$  direction and independent of the radius  $r$ , by:

$$\vec{A}(\theta, \phi) = \lim_{r \rightarrow \infty} r e^{-ik_0 r} \vec{u}_0^s(\vec{r}). \quad (8.10)$$

As before we are only going to define the cross sections for incoming plane waves. For a fluid the incoming plane wave  $\vec{u}_0^i$  is given by:

$$\vec{u}_0^i(\vec{r}) = \hat{k}_0 e^{ik_0 \cdot \vec{r}} = \frac{D}{iC} \sum_p a_p^{pl}(\alpha, \beta) \text{Re} \vec{\psi}_{3p}^0(\vec{r}) = \sum_p a_p(\alpha, \beta) \text{Re} \vec{\psi}_{3p}^0(\vec{r}). \quad (8.11)$$

Here  $a_p^{pl}$  is the same as in (8.3). We now define, for displacement fields, the differential scattering cross section  $d\sigma/d\Omega$ , the total scattering cross section  $\sigma$ , the extinction (or total) cross section  $\sigma_e$ , and the absorption cross section  $\sigma_a$  by:

$$\frac{d\sigma(\alpha, \beta, \theta, \phi)}{d\Omega(\theta, \phi)} \equiv \lim_{r \rightarrow \infty} r^2 \frac{|\vec{u}_0^s|^2}{|\vec{u}_0^i|^2} \quad (8.12)$$

$$\sigma(\alpha, \beta) \equiv \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{d\sigma}{d\Omega} d\Omega \quad (8.13)$$

$$\sigma_e(\alpha, \beta) \equiv \frac{4\pi}{k_0} \operatorname{Im} \frac{1}{D} A(\alpha, \beta, \alpha, \beta) \quad (8.14)$$

$$\sigma_a(\alpha, \beta) \equiv \sigma_e(\alpha, \beta) - \sigma(\alpha, \beta) . \quad (8.15)$$

Here  $D$  is the amplitude of the incoming plane wave. In section II we defined the relation  $\vec{u}_0 = \nabla \phi_0$ . From this relation and by comparison of equations (8.3) and (8.11) we get the relation  $D = ik_0 C$  between the amplitudes for the displacement field and scalar field. We notice that the same T-matrix relates the incoming field to the scattered field in the potential and displacement field descriptions. This fact, together with equation (8.11) and the relation  $D = ik_0 C$ , leads us to the following two relations:  $f_p = k_0 f_p^s$  and  $A = ik_0 A^s$ . From the above we can see that the cross sections defined by (8.5)-(8.8) are the same as those defined by (8.12)-(8.15).

We now turn to the case where the unbounded medium is an elastic material. In this case we have to deal with both the divergence free and curl free basis functions. (i.e.  $\tau = 1, 2$ , and 3). The expansion  $\vec{u}_0^s(\vec{r}) = \sum_q f_q \vec{\psi}_q^0(\vec{r})$  (where  $q \equiv \tau mn$ ) of the scattered displacement field  $\vec{u}_0^s$  gives us the following asymptotic behavior of the scattered field:

$$\begin{aligned} \vec{u}_0^s(\vec{r}) = \sum_p \frac{(-i)^n}{\kappa_0} \left[ \left( -if_{1p} \vec{A}_{1p}(\theta, \phi) + f_{2p} \vec{A}_{2p}(\theta, \phi) \right) \frac{e^{ik_0 r}}{r} \right. \\ \left. + \left( \frac{k_0}{\kappa_0} \right)^{1/2} f_{3p} \vec{A}_{3p}(\theta, \phi) \frac{e^{ik_0 r}}{r} \right] \text{ for large } r. \end{aligned} \quad (8.16)$$

The functions,  $\vec{A}_{\tau p}$  for  $\tau = 1, 2$  are purely transversal. The function  $\vec{A}_{3p}$  is purely longitudinal. In the case, with an unbounded elastic medium we have two kinds of incoming displacement fields. The first kind is the purely longitudinal wave given by (8.11) with a slight modification. Here, in the case with the elastic medium, we must introduce a factor  $(\kappa_0/k_0)^{3/2}$  before the expansion coefficient  $a_p$  because of a change in the normalization of the curl free basis function  $\vec{\psi}_{3p}^0$  when shifting from a fluid to an elastic medium. The second kind of incoming displacement field is the purely transversal one given by:

$$\vec{u}_0^i(\vec{r}) = \vec{E} e^{ik_0 \cdot \vec{r}} = \sum_{\tau=1,2} \sum_p a_{\tau p}(\alpha, \beta) \vec{\psi}_{\tau p}^0(\vec{r}). \quad (8.17)$$

Here  $\vec{E}$  is the amplitude vector, which is orthogonal to  $\hat{r}$ . The angles  $\alpha$  and  $\beta$  are the spherical angles for the incoming wave vector. We define, for the displacement fields, the differential and total scattering cross sections by (8.12) and (8.13), respectively. Notice that we can separate the longitudinal and the

transversal parts of the different cross sections. The total scattering cross section can, by using the relation  $\int \vec{A}_q \cdot \vec{A}_{q'} d\Omega = \delta_{qq'}$ , be expressed as:

$$\sigma(\alpha, \beta) = \frac{1}{|\vec{u}_0|^2 \kappa_0^2} \sum_p (|f_{1p}|^2 + |f_{2p}|^2 + \frac{k_0}{\kappa_0} |f_{3p}|^2) \quad (8.18)$$

## IX. NUMERICAL RESULTS

Elastic wave scattering (single scattering) by homogeneous bodies of elastic materials or cavities on an infinite elastic material is studied numerically in (12,14). Multiple scattering of elastic waves by layered bodies (several bodies) in an infinite elastic material is studied numerically in (19). Numerical results for the scattering of waves by a homogeneous body in a fluid are given in (15). For more wave number dependent data see (16). In this article we shall give some numerical results for homogeneous and layered elastic bodies in a fluid. The surfaces of the layers in the bodies are prolate spheroids with axis ratios  $a_1/b_1 = 2.0$  (see Fig. 7). The ratios  $a_2/a_1$  between the semiaxes of the inner and outer surfaces are: 0.0, 0.25, and 0.9. The inner surface always encloses a cavity. The material parameters of the fluid and of the elastic solid are given in Table 1.

We choose the z-axis as the axis of rotational symmetry. Without any loss of generality we set the spherical angle  $\beta$  for the incoming wave (see Fig. 6) equal to zero. This choice reduces the number of matrix elements needed in the finite truncation by a factor of 0.5. In the diagrams studied here we shall consider only incoming waves with the spherical angle  $\alpha = 0.0$ . In the polar plots we shall study the differential scattering cross section  $d\sigma/d\Omega$  as a function of the scattered angle  $\theta$ . In this context the other scattered angle  $\phi$  will take on the values 0.0 or 180.0. In the Cartesian plots we shall study the differential and total scattering cross sections as a function of  $k_0 a_1$ . We only consider the scattering angles  $\theta = 90$  and  $\phi = 0$  in these plots. For both the polar and Cartesian diagrams we shall consider the influence of losses in the elastic material. The complex wave numbers associated with these losses are given in Table 1.

Gauss-Legendre quadrature formulas were used to generate the matrix elements of Q, P, and R defined in Eqs. (4.12), (4.17), and (4.18). In the homogeneous case the matrix R was inverted by Gaussian elimination. However, the next inversion in Eq. (4.22) was done by Schmidt orthogonalization for the case of loss less materials. Here we used a computer routine developed by P. C. Waterman (35) wherein the symmetric and unitarity properties of the S-matrix ( $S = 1-2T$ ) have been used to optimize the inversion procedure. In the layered, loss less case we could use only the Schmidt orthogonalization process for the T-matrix of the cavity. The other inversions in this case were done by using Gaussian elimination. In the cases dealing with losses the S-matrix is not unitary, and we have to use Gaussian elimination.

The T-matrix for rotationally symmetric bodies is diagonal in the azimuth index. The two submatrices (of the T-matrix) which corresponds to the two azimuth indices 0 and 1 were given the same dimension in the finite truncation. However, for each unit increase in the value of the azimuth index beyond 1 we decreased the dimension of the associated submatrix by one. The dimension of the two biggest submatrices was progressively increased from 8 at  $k_0 a_1 = 0.1$  to 12 for  $k_0 a_1 = 3.2$ .



Figures 8-18 are plots of the cross sections, all normalized with respect to  $a_1^2$ , for a fictitious solid. The properties of this solid are given in Table 1. Figures 8-13 are polar plots. Figures 14-18 give the cross sections as functions of  $k_0 a_1$ . Figure captions give details of the individual plots. The spectra indicate that the very sharp resonances for a loss less solid almost completely disappear with the introduction of frequency dependent complex wave numbers in the solid. To obtain these sharp peaks we used a step size of 0.025 in  $k_0 a_1$ .

The method used in this paper is to our knowledge the only one working for the problems treated here. This method shares with many other methods (in other areas) the lack of a rigorous convergence proof. The parameters we have to play with in order to obtain numerical convergence are: the number of integration points and the dimension of the matrices in the finite truncation. We have made computations using the Gauss elimination and numerically tested the relations (9.1) and (9.2) for a loss less solid.

$$T^\dagger = T \quad (9.1)$$

$$T^\dagger T = - \text{Real part of } T \quad (9.2)$$

Here,  $\dagger$  means the Hermite conjugate. The discrepancy from these relations was measured relative to the maximum element of the T-matrix. We also use the ratio between the absorption cross section and the total scattering cross section as a convergence measure for the loss less solid. For homogeneous bodies our measures were usually better than  $10^{-3}$ . However, for the layered bodies they were about  $2 \cdot 10^{-2}$ . In one exceptional case, near a peak, we got as bad a measure as 0.2. It is clear that some matrices will become more illconditioned for wave numbers close to peaks. It is also clear that matrices with the combination  $OuOu$  are especially difficult to obtain with good numerical accuracy.

#### ACKNOWLEDGEMENTS

This research was supported in part by the Center for Advanced NDE operated by the Science Center, Rockwell International for the Advanced Research Projects Agency and the Air Force Materials Laboratory under Contract No. F3361-74-C-5180. V.V. Varadan and V.K. Varadan were also supported by the U.S. Office of Naval Research under Contract No. N00014-78-C-0559. B.A. Peterson was supported in part by a post doctoral fellowship from the Graduate School and the Department of Engineering Mechanics of The Ohio State University. Use of the Instructional and Research Computer Center at OSU are gratefully acknowledged. Helpful discussions with Professor Staffan Ström and Mr. Anders Boström are gratefully acknowledged. We are also grateful to Mr. Howard Tamashiro for careful reading of the manuscript.

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TABLE 1

	Water	Fict. Solid	Lossy Fict. Solid
Density $\rho_1 \text{ kg/dm}^3$	1.0	1.7	1.7
Comp. wave speed $c_p \text{ m/s}$	1493	2000	2000
shear wave speed $c_s \text{ m/s}$	zero	500	500
Comp. wave number $k_1$	$k_0$	$k_1 = 0.7465k_0$	$k_1 = 0.7465k_0[1 - ik_0 a_1 0.1]^{-1/2}$
Shear wave number $\kappa_1$	0	$\kappa_1 = 4.0k_0$	$\kappa_1 = 4.0k_0[1 - ik_0 a_1 0.1]^{-1/2}$

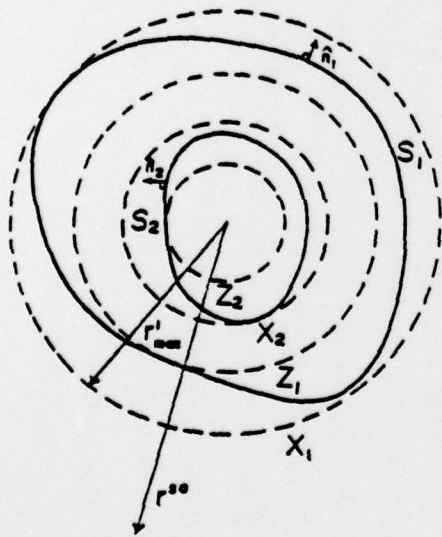
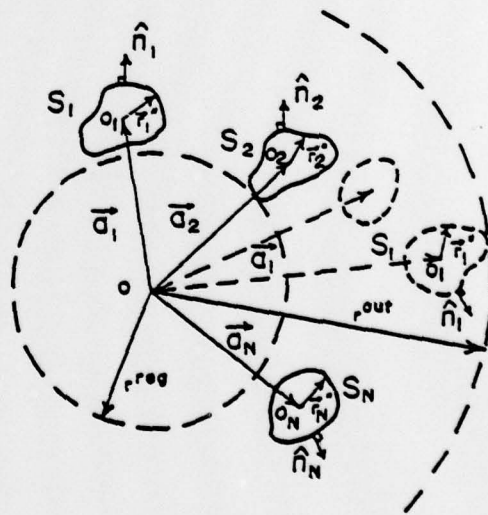


Fig. 1. Geometry of two layered body.

Fig. 2. Geometry of N scattering regions bounded by the closed surfaces  $S_i$  for  $i = 1, 2, 3 \dots N$ .

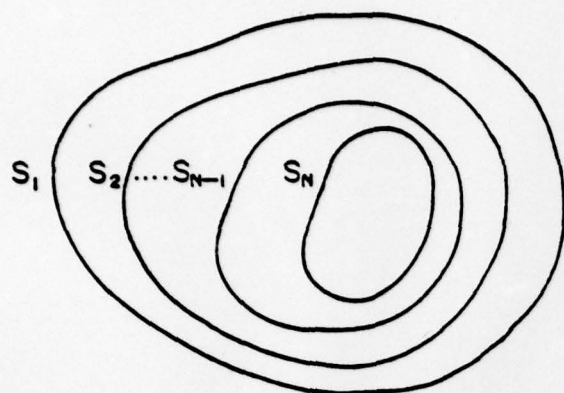


Fig. 3. Geometry of N layered body.

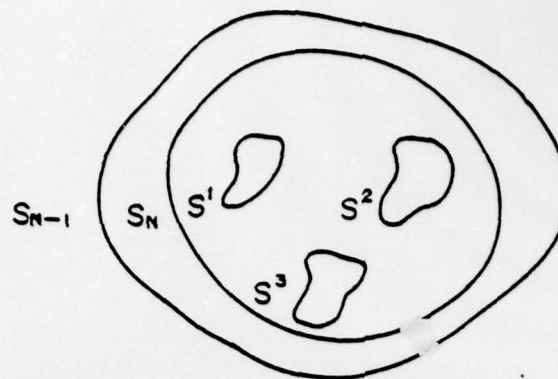


Fig. 4. Geometry of layered body with 3 scattering regions, bounded by the closed surfaces  $S_1$ ,  $S_2$ , and  $S_3$ , inside the  $N$ 'th surface.

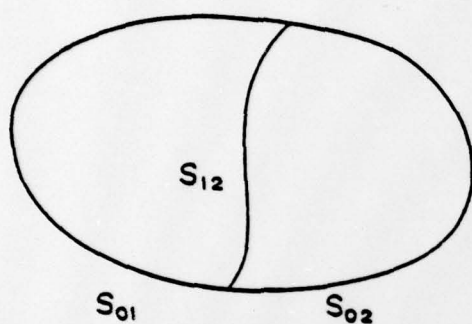


Fig. 5. Geometry of a scatterer consisting of two nonenclosing parts.

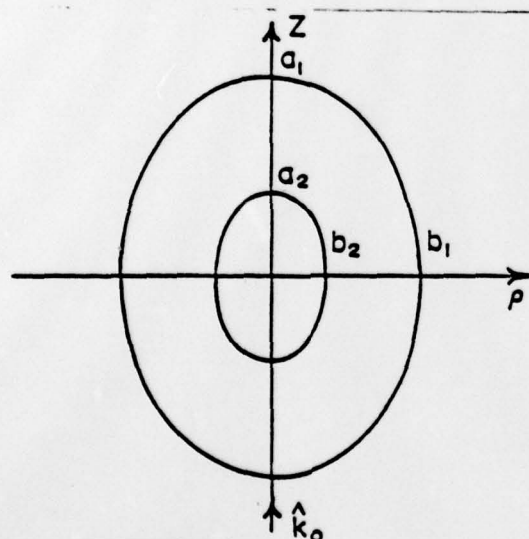


Fig. 7. Geometry of the two layered body, with two prolate spheroid surfaces, for which we give numerical results in this paper.

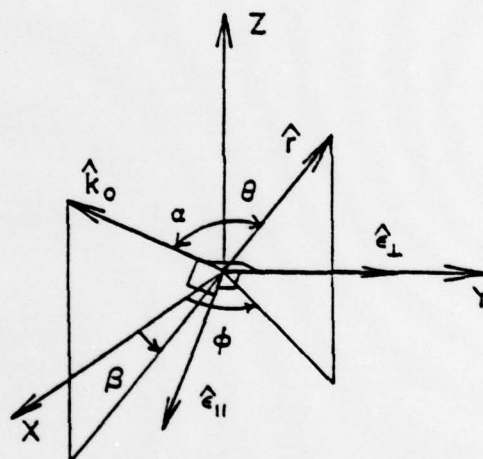


Fig. 6. Notations for incoming plane wave and scattered wave.

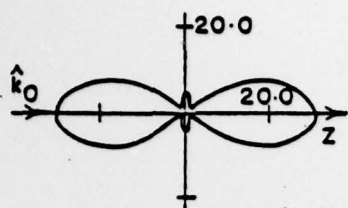


Fig. 8.a. Loss less solid.

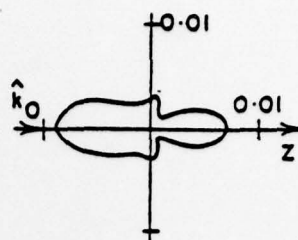


Fig. 8.b. Lossy solid.

Fig. 8. Polar plots of the differential scattering cross sections for homogeneous spheroids with  $k_0 a_1 = 1.0$ .

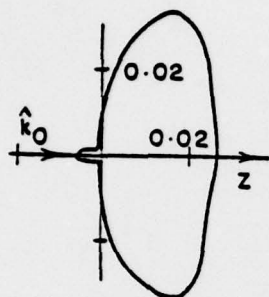


Fig. 9.a. Loss less solid.

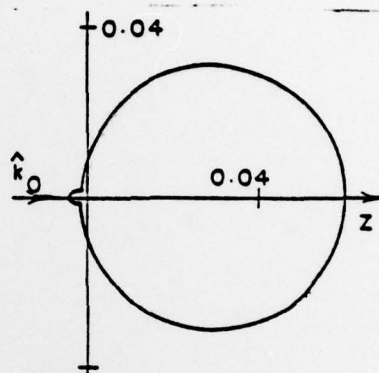


Fig. 9.b. Lossy solid.

Fig. 9. Polar plots of the differential scattering cross sections for homogeneous spheroids with  $k_0 a_1 = 3.0$ .

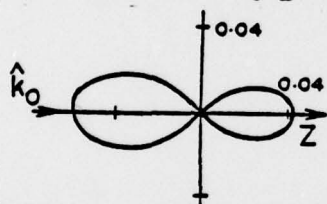


Fig. 10. Polar plot of differential scattering cross section for a layered spheroid with  $a_2/a_1 = 0.25$  and  $k_0 a_1 = 1.0$ .

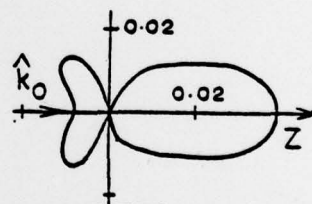


Fig. 11. Similar to Fig. 10 but here  $k_0 a_1 = 3.0$ .

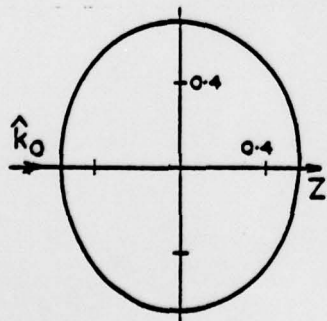


Fig. 12. Polar plot of differential scattering cross section for a layered spheroid with  $a_2/a_1 = 0.90$  and  $k_0 a_1 = 1.0$ .

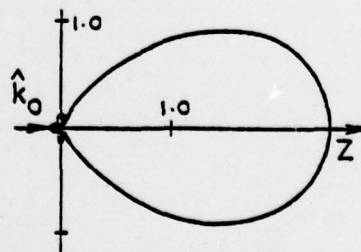


Fig. 13. Similar to Fig. 12 but here  $k_0 a_1 = 3.0$ .



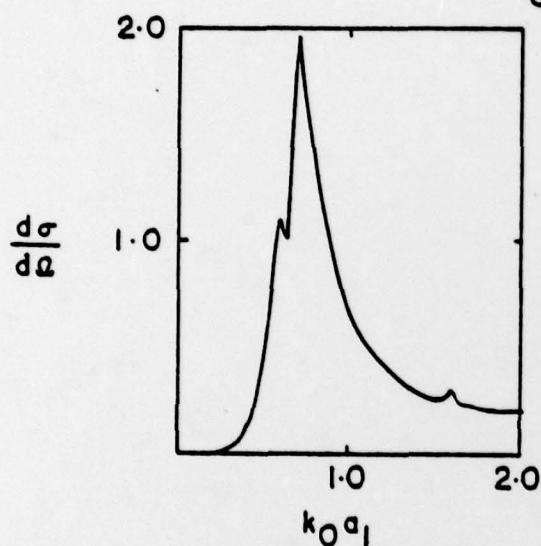


Fig. 15. Similar to Fig. 14 but here only for  $a_2/a_1 = 0.90$  loss less solid.

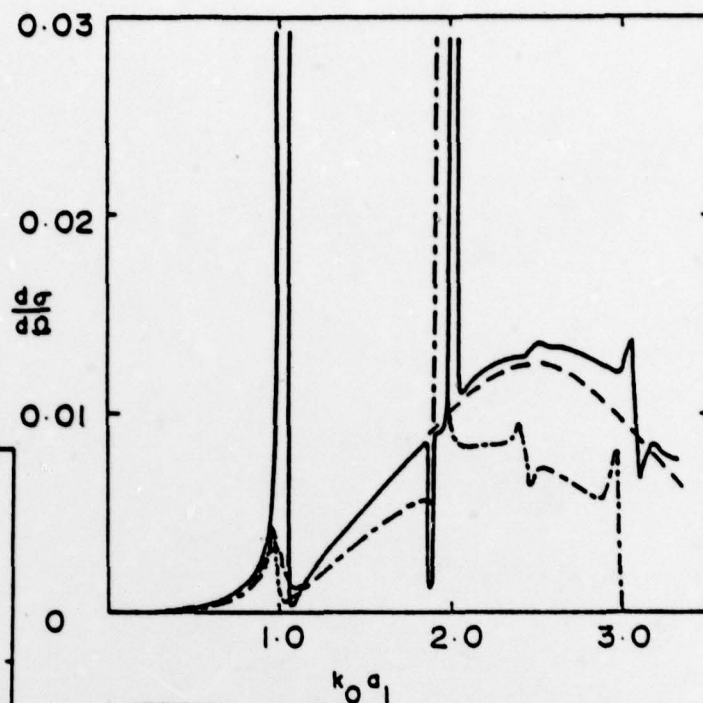


Fig. 14. Differential scattering cross section for spheroids for  $\alpha = 0$ ,  $\beta = 0$ ,  $\theta = 90$  and  $\phi = 0$ ; — for  $a_2/a_1 = 0.0$  loss less solid; - - - for  $a_2/a_1 = 0.0$  lossy solid; - · - · - for  $a_2/a_1 = 0.25$  loss less solid.

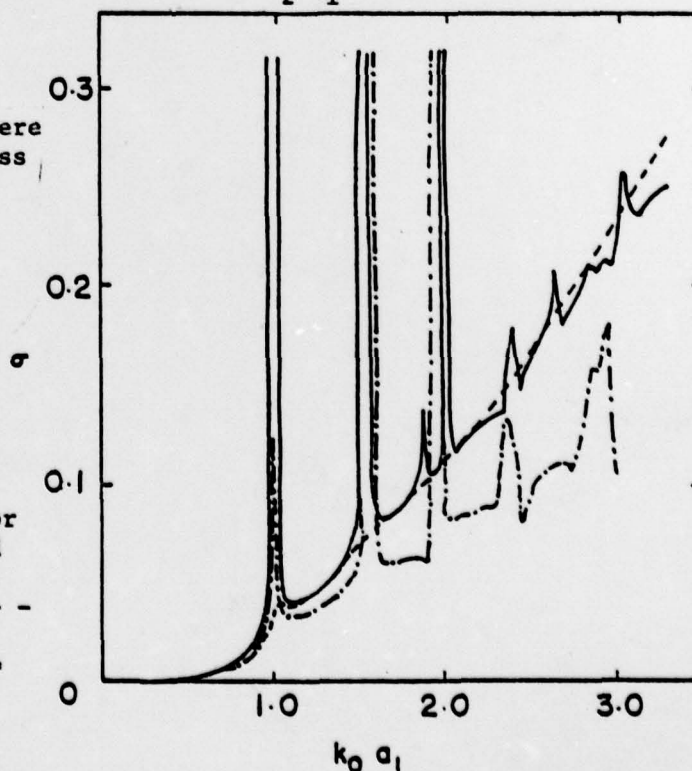


Fig. 16. Total scattering cross section for spheroids for  $\alpha = 0$ ,  $\beta = 0$ ,  $\theta = 90$  and  $\phi = 0$ ; — for  $a_2/a_1 = 0.0$  loss less solid; - - - for  $a_2/a_1 = 0.0$  lossy solid; - · - · - for  $a_2/a_1 = 0.25$  loss less solid.

Fig. 16.

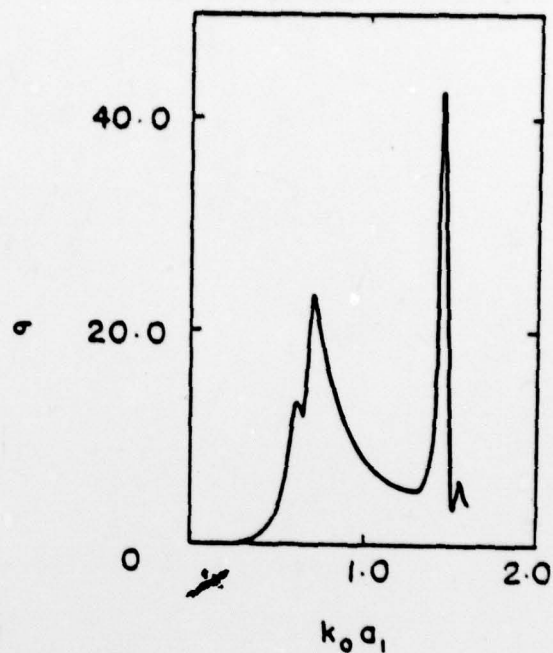


Fig. 17. Similar to Fig. 16 but here only for  $a_2/a_1 = 0.90$  and a loss less solid.

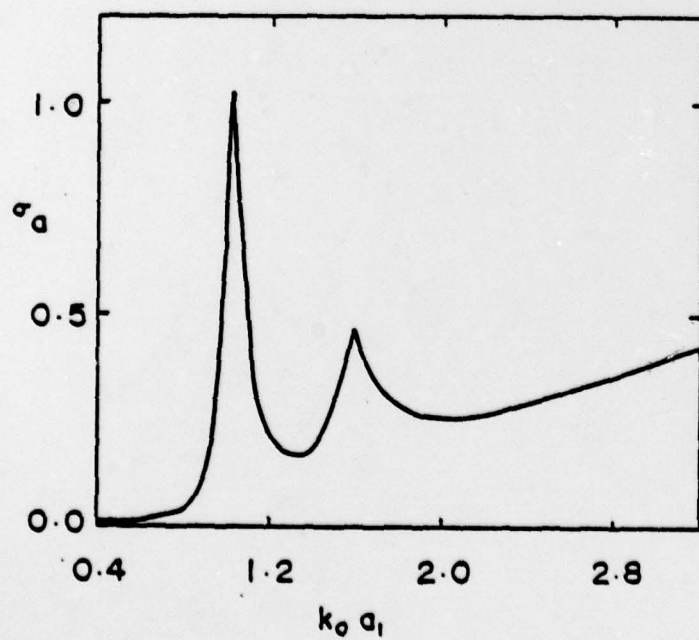


Fig. 18. Absorbtion cross section for a homogeneous spheroid for  $\alpha = 0$ ,  $\beta = 0$  and a lossy solid.